XI. *Investigations, founded on the Theory of Motion, for determining the Times of Vibration of Watch Balances. By* George Atwood, *Esq. F. R. S.*

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INSTRUMENTS for measuring time by vibratory * motion were invented early in the sixteenth † century: the single pendulum ‡ had been known to afford a very exact measure of time long before this period; yet it appears from the testimony of historical accounts, as well as other evidences, that the balance was universally adopted in the construction of the first clocks and watches; nor was it till the year 1657 that Mr. HUYGENS united pendulums with clock-work.

The first essays of an invention, formed on principles at once new and complicated, we may suppose were imperfectly executed. In the watches of the early constructions, some of

* The ancients, as early as 140 years before CHRIST (probably much earlier) were acquainted with the use of wheel-work in constructing instruments for measuring time. "Denticuli alius alium impellentes, versationes modicas faciunt ac motiones," is the expression of VITRUVIUS in describing a machine, one of the principal uses of which was to indicate the hour of the day. Vibrations are no where mentioned or alluded to in the descriptions of the clocks constructed by the ancients. Dr. DERHAM on Clock-work, p. 86, 4th edit.

† About the year 1500, according to some accounts.

‡ TYCHO BRAHE is supposed to have used the pendulum in astronomical observations. RICCIOLUS, KIRCHER, MERSENNUS, and many others, are expressly mentioned by STURMIUS to have employed this method of measuring time.

which are still preserved, the balance vibrated merely by the impulses of the wheels, without other control or regulation: the motion communicated to the balance by one impulse continued till it was destroyed, partly by friction, and partly by a succeeding impulse in the opposite direction; the vibrations must of course have been very unsteady and irregular.

These imperfections were in a great measure remedied by Dr. HOOKE'S ingenious invention of applying a spiral spring to the balance: * the action of this spring on the balance of a watch, is similar to that of gravity on a pendulum: each kind of force has the effect of correcting the irregularities of impulse and resistance, which otherwise disturb the isochronism of the vibrations.

During the present century, various improvements have been made in the construction of watches, principally by the artists of this country, to whose ingenuity and skill, aided and encouraged by public rewards, we must attribute the excellence of the modern watches and time-keepers, so highly valuable for their uses in geography, navigation, and astronomy.

The principles on which time-keepers are constructed, considered in a theoretical view, afford an interesting subject of investigation. It is always satisfactory to compare the motion of machines with the general laws of mechanics, whenever friction and other irregular forces are so far diminished as to allow of a reference to theory; especially if inferences, likely to be of practical use, may be derived from such comparison.

* Anno 1658.—An inscription on a balance-spring watch, presented to King CHARLES II. fixes the date of this invention to the year 1658. Dr. DERHAM relates, that he had seen the watch, on which the following inscription was engraved: "ROBERT HOOKE inven. 1658. T. TOMPION fecit, 1675." Dr. DERHAM on Clock-work, p. 103.

In time-keepers, the irregular forces, both of impulse and resistance, are much diminished by the exactness of form and dimension which is given to each part of the work; and they are further corrected by the maintaining power derived from the main spring: for whatever motion is lost by the balance from resistance of any kind, almost the same motion is communicated by the maintaining power, so as to continue the arc of vibration, as nearly as possible, of the same length.

In these machines, the real measure of time is the balance, all the other work serving only to continue the motion of the balance, and to indicate the time as measured by its vibrations. The regularity of a time-keeper will therefore depend on that of the time in which the balance vibrates: to investigate this time of vibration, from the several data or conditions on which it depends, is the object of the ensuing pages.

Let PMNS (Tab. XIV. fig. 1.) represent the circumference of a watch balance, which vibrates by the action of a spiral * spring, on an axis passing through the centre C. Let ODBE be the circumference of a concentric circle, considered as fixed, to which the motion of the balance may be referred. In the circumference of this circle let any point O be assumed, and when the balance is in its quiescent position, suppose a line to be drawn through C and O, intersecting the circumference of the balance in the point A; the radius CA will be an index, by which the position of the balance, and its motion through any different arcs of vibration, will be truly defined. In the ensuing pages, the motion of the balance, and the motion of the index CA, will be used indifferently, as terms conveying

* In these investigations it is indifferent whether the balance is supposed to vibrate by the action of a spiral or helical spring.

the same meaning. Since the balance is in its quiescent position when the index C A is directed to the fixed point O, on this account O is called the point of quiescence of the balance, or balance spring, indicating the position when the balance is not impelled by the spring's elastic force either in one direction or the other. If the balance should be turned through any angle O C B, the spiral spring being wound through the same angle, endeavours by its elastic force to restore itself; and when at liberty, impels the balance through the arc B O with an accelerated velocity till it arrives at the position O, where the force of acceleration ceases; with the velocity acquired at O, the balance proceeds in its vibration, describing the arc O E with a retarded motion.

The elastic forces of the spring at equal distances on the opposite sides of the point O, are assumed to be equal; it is also assumed that the effects of friction, and other irregular resistances which retard the motion of the balance, are compensated by the maintaining power, so that the time of describing the first arc of vibration B O by an accelerated motion, shall be equal to the time of describing the latter arc O E by a retarded motion, and that the entire arc of vibration B O E is bisected by the point O.

To render the construction of fig. 1. more distinct, the fixed circle O D B E is represented to be at a small distance from the circumference of the balance, but is to be considered as coincident with it, so that the arc B O subtending the angle B C O, may be of the same length with an arc of the circumference of the balance which subtends the same angle B C O: on this principle C O or C A may be taken indifferently as the radius of the balance.

The determination of the time in which the balance vibrates, from the theory of motion, requires the following particulars to be known.

1st. The spring's elastic force, which impels the circumference of the balance when it is at a given angular distance O D (fig. 1.) from the quiescent point O.

2dly. The law or ratio observed in the variation of the spring's force, while the balance is impelled from the extremity of the semiarc B to the point of quiescence O, where all acceleration ceases.

3dly. The weight of the balance, including the parts which vibrate with it.

4thly. The radius of the balance C O, and the distance of the centre of gyration from the axis of motion C G.

5thly. The length of the semiarc B O.

Suppose the plane of the balance to be placed vertically, and let a weight P (fig. 2.) be applied by means of a line suspended freely from the circumference at T, to counterpoise the elastic force of the spring when the balance is wound through an angle from quiescence O C D. This weight P (the weight of the line being allowed for) will be the force of the spiral spring which impels the circumference of the balance, when at the angular distance O D, from the quiescent position.

It appears from many experiments, that the weights necessary to counterpoise a spiral spring's elastic force, when the balance is wound to the several distances from the quiescent point, represented* by the arcs OG, OH, OI, (fig. 2.) &c. are nearly in the ratio of those several arcs. It also appears, that the shape, the length, and number of turns of the spiral may be so adjusted

* Berthoud *Traité des Horloges marines*, p. 49.

to each other, that the forces of elasticity shall be counterpoised by weights which are in the precise ratio of the angular distances from the quiescent position, or, as it is sometimes expressed, in the ratio of the spring's tensions; at least as nearly as can be ascertained by experiment: this law of elastic force is assumed in the subsequent investigation.

The position of the centre of gyration may be always determined when the figure of the vibrating body is regular, by calculating the sum of the products which arise from multiplying each particle into the square of its distance from the axis of motion, and dividing the sum by the weight of the vibrating body; the square root of the result will be the distance of the centre of gyration from the axis of motion. When the figure of the vibrating body is irregular, recourse may be had to experimental * methods, in order to determine the position of the centre of gyration.

Let the radius of the balance CA or CO $= r$, (fig. 1.) the semiarc BO $= b$; let the spring's elastic force, acting on the circumference of the balance, when wound to any given angle OCD from the quiescent position be $=$ P, and let the arc OD $= a$; the weight of the balance, and the parts which vibrate with it $=$ W; the distance of the centre of gyration from the axis of motion CG $= g$. These notations being premised, the resistance of inertia by which the mass contained in the balance opposes the communication of motion to the circumference is $\frac{Wg^2}{r^2}$: and consequently the force which accelerates the circumference at the angular distance OCD from the quiescent position is $\frac{Pr^2}{Wg^2}$. This quantity remaining invariably the same, while the balance describes the arc of vibration BOE, may be denoted by the letter F, so that

* Treatise on the Rectilinear Motion and Rotation of Bodies, p. 226 and 301.

$F = \frac{P r^2}{W g^2}$; suppose the radius CA commencing a vibration from the point B to have described the arc B H, and let O H $= x$; since the force which accelerates the circumference at the angular distance from quiescence O D is $=$ F, and the forces of acceleration are supposed to vary in the proportion of the angular distances from the quiescent point O, the force which accelerates the circumference of the balance at the point H will be $= \frac{Fx}{a}$; let u be the space through which a body falls freely from rest by the acceleration of gravity to acquire the velocity of the circumference, at the point H; the principles of acceleration give this equation, * $\dot{u} = \frac{-Fx\dot{x}}{a}$;

* Newtonii Princ. Vol. I. prop. xxxix. Let a body describe the line A C by the acceleration of a force varying in any ratio of the distances from a centre C. Let another body describe the line E H by the acceleration of a constant or uniform force. Suppose the velocity at O to be equal to the velocity at D, and let O G and D F be the evanescent spaces, or increments of space in which equal velocities are generated; so that E D may represent a line through which a body must fall from rest by the acceleration of the constant or uniform force, to acquire the velocity of the other body at O. It is to be proved that the increment of space O G is to the increment of space D F, as the force of acceleration at D to the force of acceleration at O. Let the former of these forces, *i. e.* at D be denoted by G, and the latter force at O by H. Let E D $= u$, and let A O $= x$. Also let D F $= \dot{u}$, and O G $= \dot{x}$.

Because the increments of velocity are always as the forces of acceleration and the elementary times in which they act jointly, it follows, that when the increments of velocity are equal, the forces are in the inverse ratio of the elementary times in which they act; that is (the velocities of describing the evanescent spaces O G, D F being equal by the supposition), the forces are in the inverse ratio of those spaces; and consequently the force at D (G) is to the force at O (H) as O G to D F; that is, according to the preceding notation, G : H : : $\dot{x} : \dot{u}$ or $\dot{u} = \frac{H\dot{x}}{G}$. The constant force G being assumed equal to that of gravity, may be denoted by any constant quantity, such as unity. By substituting therefore 1 for G, the equation will become $\dot{u} = H\dot{x}$. In this

and taking the fluents while x decreases from b to x, $u = \frac{F \times \overline{b^2 - x^2}}{2a}$: if therefore l is made $= 193$ inches, being the space which bodies falling freely from rest by the force of gravity near the earth's surface describe in one second of time, the velocity of the circumference, when the extremity A of the index C A has arrived at the point H, will be

$= \sqrt{\frac{2lF}{a}} \times \sqrt{b^2 - x^2}$.

Let t represent the time in which the circumference describes the arc B H; then will $\dot{t} = \sqrt{\frac{a}{2lF}} \times \frac{-\dot{x}}{\sqrt{b^2 - x^2}}$; and $t = \sqrt{\frac{a}{2lF}} \times$ into a circular arc, of which the cosine $= \frac{x}{b}$ to radius $= 1$, which is the time of describing the arc B H expressed in parts of a second; when $x = 0$, that is when the circumference has described the entire semiarc B O, the circular arc of which the cosine $= \frac{x}{b}$ is a quadrant of a circle to radius $= 1$. Let $p = 3.14159$, &c. The time t, of describing the semiarc B O $= \sqrt{\frac{a}{2lF}} \times \frac{p}{2} = \sqrt{\frac{p^2 a}{8lF}}$.

In this expression for the time of a semivibration, the letter a denotes the length of the arc OD (fig. 1.); if this arc should be expressed by a number of degrees c°, a will then $= \frac{prc^\circ}{180^\circ}$; and this quantity being substituted for a, the time of a semivibration will be $t = \sqrt{\frac{p^3 r c^\circ}{8lF \times 180^\circ}}$; if instead of F, its value $\frac{Pr^2}{Wg^2}$ is substituted in the equation $t = \sqrt{\frac{p^3 r c^\circ}{8lF \times 180^\circ}}$, the time of a semivibration will be $t = \sqrt{\frac{Wp^3 g^2 c^\circ}{8Prl \times 180^\circ}}$.

Let the given arc c° be $= 90^\circ$; in this case $t = \sqrt{\frac{Wp^3 g^2}{16Prl}}$.

expression the lines u and x are supposed to increase together; but if u increases while x decreases, the signs of the variations $\dot{u}$ and $\dot{x}$ will be contrary; in which case the equation will become $\dot{u} = -H\dot{x}$.

These are expressions for the time of a semivibration, whatever may be the figure of the balance, the other conditions remaining the same as they have been above stated. If the balance should be a cylindrical plate, it is known that the distance of the centre of gyration from the axis is to the radius as 1 to $\sqrt{2}$; wherefore in this case $g^2 = \frac{r^2}{2}$; and the time of a semivibration, or $t = \sqrt{\frac{W p^3 r}{32 P l}}$.*

* The balances of watches are usually of such a form as to place the centre of gyration nearly at the same distance from the axis, as if the figures were cylindrical plates of uniform thickness and density. If it should be required to obtain from theory the time of a balance's vibration precisely exact, it would be necessary to calculate rigidly the position of the centre of gyration from the dimensions of each part of the balance, and whatever vibrates with it. But in cases merely illustrative of the general theorems for ascertaining the times of vibration, it is unnecessary to enter into prolix and troublesome calculations depending on the form of any particular balance; since by assuming it as a cylindrical plate, the time of a vibration will not differ materially from that which would be the result of the correct investigation.

Being desirous of comparing the time of vibration, as deduced from the theory of motion, with the actual vibration of a watch balance, I requested Mr. EARNSHAW (the excellent performance of whose time-keepers is well known) to make the experiments from which the necessary data for this calculation are derived. These experiments were made on the balance of a watch constructed by Mr. KENDAL, on Mr. HARRISON's principles, and is the instrument which Captain COOK took out with him during his last voyage to the South Seas. The results are underneath:

Diameter of the balance - - - - - - $2\frac{1}{4}$ inches.

Weight of the balance, and parts which vibrate with it - - 42 grains.

Weight applied to the circumference of the balance, which counterpoises the force of the spiral spring when the balance is wound through an angle of 180° - - - - - - 48 grains.

The weight which counterpoises the spring's force when the balance is wound to 90 degrees from quiescence is - - - 24 grains.

These determinations give the following substitutions in the expression for the time of a semivibration $t = \sqrt{\frac{W p^3 r}{32 P l}}$.

It is observable that the semiarc of vibration $BO = b$, does not enter into these expressions for the time of a semivibration; if therefore instead of the semiarc B O, an arc of any other length LO, terminating in the point of quiescence O, (fig. 1.) should be substituted in the preceding investigation, the time of describing L O would be still $= \sqrt{\frac{ap^2}{8lF}}$ or $\sqrt{\frac{p^3 r c^\circ}{8lF \times 180^\circ}}$ equal to the time of describing the other semiarc B O; consequently, whether the balance vibrates in the largest or smallest arcs, the times of vibration will be the same.

From the preceding investigations it appears, that when the force by which the circumference of the balance is accelerated at the given angular distance c° from the quiescent position is $= F$, the time of a semivibration $t = \sqrt{\frac{p^3 r c^\circ}{8lF \times 180^\circ}}$; and conversely, when the time of a semivibration is $= t$, the force which accelerates the circumference at the given angular distance c° from the quiescent position, that is $F = \frac{p^3 r c^\circ}{8lt^2 \times 180^\circ}$.

Since watches and time-keepers are usually adjusted to

Namely, $W = 42$ gr. = the weight of the balance, including the parts which vibrate with it.

$P = 24$ gr. = the force at the circumference of the balance, which counterpoises the force of the spring when wound to the distance 90°.

$r = 1.125$ inch. and parts = the radius of the balance.

$l = 193$ inches = the space described in one second of time by bodies which descend freely from rest by the acceleration of gravity.

$p = 3.14159$, &c. = the circumference of a circle of which the diameter $= 1$;

	pts. of a second.
the time of a semivibration $t = \sqrt{\frac{42 \times \overline{3.14159}^3 \times 1.125}{32 \times 24 \times 193}} =$	0.0994
The balance, when adjusted to mean time, makes 5 vibrations in a second; the actual time of a semivibration is therefore - -	0.1000
Difference between the actual time and the time by the calculation -	0.0006

mean time when the balance makes 5 vibrations in a second, the time of a semivibration will in this case $= \frac{1}{10}$ part of a second: the substitution of $\frac{1}{10}$ for t being made in the preceding equation, the force which accelerates the circumference of the balance, when at any given angular distance $c°$ from the quiescent position, will be determined for all time-keepers adjusted to mean time, in which the balances make 5 vibrations in a second. Suppose the given angle $c° = 90°$; then making $c° = 90°$, $p = 3.14159$, &c. $l = 193$, $t = \frac{1}{10}$, the accelerative force at the angular distance from quiescence $90°$ or $F = \frac{p^3 \, r \, 90°}{8 \, l \, t^2 \times 180°} = r \times 1.00408926$. We have therefore arrived at the following conclusion: if the radius of the balance is equal to 1 inch, and the time-keeper is adjusted to mean time when the balance makes 5 vibrations in a second, the force which accelerates the circumference of the balance at the distance of 90° from the quiescent position, is $= 1.00408926$, the accelerative force of gravity being $= 1$. And if the radius of the balance is greater or less than 1 inch, the force by which the circumference is accelerated at the distance of 90° from quiescence, will be greater or less than 1.00408926 in proportion to the radii.

According to the principles assumed in the preceding solution, the spring's elastic force is supposed to vary in the proportion of the angular distances from the quiescent position, and on this condition, the vibrations are shewn to be isochronous, whether they are performed in longer or shorter arcs; but if the spring's elastic force at different distances from quiescence should not be precisely in the ratio here assumed, the longer and shorter arcs may be described in times differing in any proportions of inequality. If, for instance, the spring's

force, instead of varying in the ratio of the aforesaid distances, should vary in the $\frac{999}{1000}$ power, or $\frac{1001}{1000}$ power of the distances, it does not appear from the preceding solution what alteration in the daily rate would be caused by this change in the law of the force's variation, when the semiarc of vibration is increased or diminished by a given arc. To ascertain this point fully, other researches will be necessary, by which it may be known, what alteration of the daily rate of a time-keeper is occasioned by a given increase or diminution of the arc of vibration, when the spring's elastic force varies in a ratio of the distances from the quiescent position, the general index or exponent of which is any number or fraction n.

The force which accelerates the balance being assumed in that power of the distances the exponent of which is n, let $BO = b$ (fig. 3.) be the semiarc of vibration when the time-keeper is adjusted to mean time; let $DO = a$; the accelerating force on the circumference at the distance from quiescence $OD = F$; suppose the circumference to have described the arc BH from the extremity of the arc B; and let $HO = x$: then the force by which the circumference is accelerated when at the angular distance from the quiescent position $OH = \frac{Fx^n}{a^n}$; let u be the space through which a body falls freely from rest by the acceleration of gravity, to acquire the velocity of the circumference when it has described the arc BH; the principles of acceleration give this equation:

$\dot{u} = \frac{-Fx^n\dot{x}}{a^n}$: taking the fluents while x decreases from b to x,

$u = \frac{Fb^{n+1} - Fx^{n+1}}{\overline{n+1} \times a^n}$, and l being 193 inches, the velocity acquired by the circumference after describing BH, will be

$$=\sqrt{\frac{4lF}{\overline{n+1}\times a^n}}\times\sqrt{b^{n+1}-x^{n+1}};$$

let T be the time of describing the arc BH; wherefore

$$\dot{T}=\sqrt{\frac{\overline{n+1}\times a^n}{4lF}}\times\frac{-\dot{x}}{\sqrt{b^{n+1}-x^{n+1}}}.$$

The time of describing the arc BH will be the fluent of this fluxion, while x decreases from b to x, and the time of describing the semiarc BO will be the entire fluent of

$\sqrt{\frac{\overline{n+1}\times a^n}{4lF}}\times\frac{-\dot{x}}{\sqrt{b^{n+1}-x^{n+1}}}$, while x decreases from b to o.

Now let the balance commence its vibration from any other point I, (fig. 3.) and let IO $=c$; suppose the circumference to have described the arc IK, and make OK $=y$; let t be the time of describing the arc IK; then by proceeding in the same manner as in the former case, it is found that $\dot{t}=\sqrt{\frac{\overline{n+1}\times a^n}{4lF}}$ $\times\frac{-\dot{y}}{\sqrt{c^{n+1}-y^{n+1}}}$; and the time of describing the semiarc IO, will be the entire fluent of this fluxion, while y decreases from c to o. Although the fluents of the fluxions $\frac{-\dot{x}}{\sqrt{b^{n+1}-x^{n+1}}}$ and $\frac{-\dot{y}}{\sqrt{c^{n+1}-y^{n+1}}}$ cannot be expressed in general terms, yet the exact proportion of the said fluents may be assigned, which will be the proportion of the times in which the balance vibrates in the two semiarcs BO, IO; the multiplying quantity $\sqrt{\frac{\overline{n+1}\times a^n}{4lF}}$ being common to both fluxions; and since the entire fluent* of $\frac{-\dot{x}}{\sqrt{b^{n+1}-x^{n+1}}}$ is to the entire fluent of

* $\frac{\dot{x}}{\sqrt{b^{n+1}-x^{n+1}}}=\frac{1}{b^{\frac{n+1}{2}}}\times\frac{\dot{x}}{\sqrt{1-\overline{\frac{x}{b}}\vert^{n+1}}}$

$\frac{-\dot{y}}{\sqrt{c^{n+1}-y^{n+1}}}$ as $b^{\frac{1-n}{2}}$ is to $c^{\frac{1-n}{2}}$, it follows, that the time of a semivibration in the arc B O is to the time of a semivibration in the arc I O, as $b^{\frac{1-n}{2}}$ to $c^{\frac{1-n}{2}}$, or as 1 to $\overline{\frac{IO}{BO}}\Big|^{\frac{1-n}{2}}$.

Suppose a watch to be adjusted to mean time when the semiarc of the balance's vibration is = B O (fig. 3.) and let this semiarc be afterwards diminished to I O; the time shewn by the watch in any given portion of mean time t, when the semiarc of vibration is I O, will be $= t \times \overline{\frac{BO}{IO}}\Big|^{\frac{1-n}{2}}$; and if t is put $= 24^{h}$, the alteration of the daily rate, in consequence of the dimi-

and $\frac{\dot{y}}{\sqrt{c^{n+1}-y^{n+1}}} = \frac{1}{c^{\frac{n+1}{2}}} \times \frac{\dot{y}}{\sqrt{1-\overline{\frac{y}{c}}\big|^{n+1}}}$.

To find the proportion of the entire fluent of $\frac{\dot{x}}{\sqrt{1-\overline{\frac{x}{b}}\big|^{n+1}}}$

to the entire fluent of $\frac{\dot{y}}{\sqrt{1-\overline{\frac{y}{c}}\big|^{n+1}}}$

make $x = \frac{by}{c}$, so that when $x = 0$, $y = 0$, and when $x = b$, $y = c$;

then $\frac{\dot{x}}{\sqrt{1-\overline{\frac{x}{b}}\big|^{n+1}}} = \frac{b}{c} \times \frac{\dot{y}}{\sqrt{1-\overline{\frac{y}{c}}\big|^{n+1}}}$,

and the proportion of $\frac{\dot{x}}{\sqrt{1-\overline{\frac{x}{b}}\big|^{n+1}}}$ to $\frac{\dot{y}}{\sqrt{1-\overline{\frac{y}{c}}\big|^{n+1}}}$ will be equal to that of $\frac{b}{c}$ to 1, or of b to c; this being the constant proportion of the fluxions when $x = \frac{by}{c}$; the fluents will be in the same proportion, provided $x = \frac{by}{c}$; wherefore the entire fluent of $\frac{\dot{x}}{\sqrt{b^{n+1}-x^{n+1}}}$ will be to the entire fluent of $\frac{\dot{y}}{\sqrt{c^{n+1}-y^{n+1}}}$ as $\frac{b}{b^{\frac{n+1}{2}}}$ to $\frac{c}{c^{\frac{n+1}{2}}}$; or as $b^{\frac{1-n}{2}}$ to $c^{\frac{1-n}{2}}$.

nution of the semiarc of vibration from B O to I O, will be $24^h \times \overline{\frac{BO}{IO}}\Big|^{\frac{1-n}{2}} - 1$.*

To apply this proposition, let a case be assumed; suppose a watch to be regulated to mean time when the semiarc of vibration is 135°, and let this semiarc be diminished 8°, so as to become 127°; let the ratio of the spring's elastic

* From this general expression it appears, that when $n = 1$, that is when the spring's elastic force varies in the precise ratio of the angular distances of the balance from the quiescent position, the alteration of the daily rate in consequence of a diminution of the arc of vibration is $= 0$; because in that case $\overline{\frac{BO}{IO}}\Big|^{\frac{1-n}{2}} = 1$, and $\overline{\frac{BO}{IO}}\Big|^{\frac{1-n}{2}} - 1 = 0$. When n is less than 1, or when the force varies in a less ratio than that of the distances from quiescence, the rate will be accelerated, because in that case $\overline{\frac{BO}{IO}}\Big|^{\frac{1-n}{2}}$ will be greater than 1; and $\overline{\frac{BO}{IO}}\Big|^{\frac{1-n}{2}} - 1$, will be a positive quantity: but when n is greater than 1, or when the force varies in a ratio greater than that of the distances from quiescence, the rate will be retarded, because in that case $\overline{\frac{BO}{IO}}\Big|^{\frac{1-n}{2}}$ will be less than 1, and $\overline{\frac{BO}{IO}}\Big|^{\frac{1-n}{2}} - 1$ becomes negative. The converse of these propositions is likewise derived from the general theorem.

Whenever therefore it is found, by observing the rate of a time-keeper, that a diminution of the arc of the balance's vibration causes an acceleration of the daily rate, it is necessary to conclude, that the elastic force of the spring in this case varies in a ratio less than that of the distances from the quiescent position. In like manner, when a diminution of the arc of vibration causes a retardation of the rate, it is certain that the spring's elastic force varies in a higher ratio than that of the distances from quiescence. It appears, indeed, from some experiments, that the weights which counterpoise a spiral spring's elastic force, when wound to different distances from the quiescent position, are in the ratio of those distances; but it is shewn from this proposition, and the annexed table, that the differences between the weights, by which the ratio of the distances, and a ratio a little less is indicated, although far too small to be discoverable by experiment, are yet sufficient to create a material alteration of the daily rate.

force deviate from that of the distances from the quiescent position by a small difference of $\frac{1}{1000}$ part, so that the spring's force shall be in the $\frac{999}{1000}$ power of the distances, instead of in the entire ratio of the said distances from the quiescent position. The alteration in the daily rate of the watch will be obtained from the preceding theorem by making the following substitutions. $BO = 135^\circ$, $IO = 127^\circ$, $n = \frac{999}{1000}$: the alteration of the daily rate $= 24^h \times \overline{\frac{135}{127}}\Big|^{\frac{1}{2000}} - 1 =$ $+ 2''.62$.

It here appears, that a very minute alteration in the law of the force's variation, amounting to no more than $\frac{1}{1000}$ part of the entire ratio of the distances, causes an acceleration in the daily rate of more than $2\frac{1}{2}''$, when the diminution of the semiarc of vibration is 8°. It may therefore be of some use to inquire, what are the differences of the weights to be observed in experiments from which the law of the spring's elastic force is derived; first, supposing that law to be the precise ratio of the distances from the quiescent position; and secondly, supposing the law of the force to deviate from that ratio by a small difference of $\frac{1}{1000}$, so as to become the $\frac{999}{1000}$ power of the distances from the quiescent position; from the result a judgment may be formed how far experiments may be relied on for ascertaining the precise law according to which the elastic force of a spring varies.

Values of x°.	Values of $P = 9\text{gr.} \times \frac{x^{\circ}}{90^{\circ}}$	Values of $P = 9\text{gr.} \times \overline{\frac{x^{\circ}}{90^{\circ}}}\Big\vert^{\frac{999}{1000}}$
Angular distances from the quiescent position when the spring's elastic force is counterpoised by the weights in the second and third columns. (Fig. 2).	Weights P, expressed in grains, which counterpoise the spring's elastic force when wound to the several distances from the quiescent position in the first column, if the force varies in the precise ratio of the angular distances from the quiescent point.	Weights P, expressed in grains, which counterpoise the spring's elastic force when wound to the several distances from quiescence in the first column, if the force varies in the $\frac{999}{1000}$ power of the distances from the quiescent point.*
	Grains.	Grains.
10°	1	1.002199
20°	2	2.003010
30°	3	3.003298
40°	4	4.003245
50°	5	5.002940
60°	6	6.002433
70°	7	7.001759
80°	8	8.000942
90°	9	9.000000

The differences of weights expressed in the second and third columns of this table are evidently too small to admit of being observed experimentally, and yet their effect on the daily rate of a time-keeper amounts to a quantity far from insensible. This effect on the rate might probably be augmented to twenty or thirty seconds daily, and yet the corresponding differences of weights arising from the deviation of the spring's force from the law of isochronism might be too minute to become sensible by any statical counterpoise of the spring's forces; and it would

* It may be here observed, that the differences of the weights in the second and third columns of the table, first increase and afterwards decrease; their difference is the greatest when the quantity $9 \times \overline{\frac{x}{90}}\Big\vert^{\frac{999}{1000}} - \frac{9x}{90}$ is a maximum; or when $x = 90^{\circ} \times \overline{\frac{999}{1000}}\Big\vert^{1000}$, that is when $x = 33^{\circ}\ 5'\ 33''$.

be still less possible to measure the said differences of weights with the exactness required for the determination of the law observed by the spring's forces. Experiments of this kind should not therefore be absolutely relied on for ascertaining practically the isochronal property of spiral springs, although this property must be allowed to exist in theory, whenever the forces of elasticity at the several angular distances from the quiescent position are in the precise ratio of those distances. The isochronal law of variation, here mentioned, may be conveniently assumed in theoretical investigations, and proper corrections or equations, when necessary, may be applied to compensate for the deviation from this law, which may subsist in any particular spiral spring, whenever it can be satisfactorily ascertained, or reduced within known limits, by such mode of inference as the nature of the case may admit of. This assumption will appear the less exceptionable from considering, that the elastic forces of spiral springs which are not isochronal deviate from the law of variation in question, in some cases by exceeding, and in others, by falling short of it; and no other law is suggested either by theory or experiment, which more generally corresponds with the action of balance springs.

The vibration of a balance impelled by a single spiral spring only, has been the subject of the preceding investigations; but cases occur in which two or more springs are employed in giving vibratory motion to the balances of watches. Not to mention preceding instances, Mr. MUDGE, an eminent watch-maker of the present times, has invented a method of combining the action of spiral springs, to impel the balance in each semiarc of vibration, on a principle not more remarkable for the novelty than it is for the ingenuity of the contrivance. The consideration of this additional case will therefore not be

thought foreign to the present subject, especially as it may contribute to elucidate some circumstances respecting the effect of springs on the vibrations of balances, which at the first view are not at all obvious.

Let two spiral springs be applied to act on a watch balance in the same direction; if the two springs in unwinding themselves, by turning the balance come to the same point of quiescence, or, in other words, if the accelerative forces of both springs cease at the middle point of the vibration, whatever be the relative strength of the two springs, they will act on the balance precisely in the same manner as if one spring only had been applied, of equal strength with both (the springs being here supposed similar, in respect of the law of the elastic forces and tensions). But when the points of quiescence of the two springs do not coincide, that is, when one spring continues to accelerate the balance in its vibration, after the acceleration of the other spring has ceased, the time of a semivibration must be obtained from a separate investigation.

Let the circumference of a balance (fig. 4.) be impelled by the action of a spiral spring through the semiarc of vibration BO, the forces of this spring being always in proportion to the angular distances from the point of quiescence O; let a secondary or auxiliary spring also act on the balance from the extremity of the semiarc B as far as the point Q, at which point all acceleration of the auxiliary spring ceases, the forces of the auxiliary spring varying as the angular distances from the point of quiescence Q. Suppose that the accelerative force of the principal or balance spring on the circumference of the balance at the distance from quiescence OD is $= f$, take $Q\,d = O\,D$, and let the accelerative force of the balance spring on the circumference of the balance be to that of the auxiliary spring, when both springs are wound to

the same angle $OCD = QCd$, in the proportion of 1 to n; then the accelerative force of the auxiliary spring at the angular distance from quiescence Qd will $= nf$; let $BO = b$, $BQ = c$, $QO = d = b - c$; also let OD or $Qd = a$.

Suppose the balance to have described the arc BH by the joint action of both the springs, and let the arc BH be represented by x. Then, because the accelerative force of the principal or balance spring at the angular distance from quiescence $OD = a$ is f, the accelerative force of the same spring at the distance OH is $= \frac{f \times \overline{b-x}}{a}$; and since the force of the auxiliary spring at the angular distance from quiescence $Qd = a$ is nf, the accelerative force of this spring at the angular distance from quiescence QH will be $= \frac{nf \times \overline{c-x}}{a}$; wherefore the joint force of both springs to accelerate the circumference when at the distance OH from the point of quiescence O, that is, when the balance has described the arc BH, will be $= \frac{f}{a} \times \overline{b - x + cn - nx}$. Let u be the space through which a body falls freely from rest by the acceleration of gravity, to acquire the velocity of the circumference when it has described the arc BH; this will give the following equation:

$\dot{u} = \frac{f}{a} \times \overline{b\dot{x} - x\dot{x} + cn\dot{x} - nx\dot{x}}$; and

$u = \frac{f}{2a} \times \overline{2bx - x^2 + 2cnx - nx^2}$; and if $l = 193$ inches, the velocity of the circumference when the arc BH has been described, is $= \sqrt{\frac{2lf}{a}} \times \sqrt{2bx - x^2 + 2cnx - nx^2}$. Let t represent the time in which the balance describes the arc BH; then $\dot{t} = \sqrt{\frac{a}{2lf}} \times \frac{\dot{x}}{\sqrt{2bx - x^2 + 2cnx - nx^2}}$ and the fluent or $t = \sqrt{\frac{2a}{lf \times \overline{n+1}}} \times$ into a circular arc of which the sine is

$\sqrt{\frac{x \times \overline{n+1}}{2b+2nc}}$ to radius $= 1$; which is the time of describing the arc B H. When $x = c$, the time of describing the arc B Q, is $\sqrt{\frac{2a}{lf \times \overline{n+1}}} \times$ into a circular arc of which the sine $=$ $\sqrt{\frac{c \times \overline{n+1}}{2b+2nc}}$.

The time of describing the remaining part of the semiarc Q O, (fig. 4.) is next to be determined. In the cases to which this investigation is applied, the auxiliary spring ceases entirely to act on the balance after it has described the arc B Q. This being stated, while the balance describes the remaining arc of the semivibration Q O, it will be impelled by the balance spring only. To ascertain the time of describing the arc Q O, it is first to be observed, that when the circumference has described the arc B Q, it will have acquired a velocity equal to that of a body which has fallen from rest by the acceleration of gravity through a space* $= \frac{f}{2a} \times \overline{2bc - c^2 + nc^2}$.

Suppose the balance to have proceeded through the arc Q R, (fig. 4.) and let O R $= x$; the force by which the circumference is accelerated at R $= \frac{fx}{a}$; and if u is the space through which a body falls freely from rest by the acceleration of gravity to acquire the velocity of the circumference in the point R, $\dot{u} = \frac{-f\dot{x}x}{a}$; taking the fluents so that u may become $=$ $\frac{f}{2a} \times \overline{2bc - c^2 + nc^2}$ when $x = d$, $u = \frac{f}{2a} \times \overline{2bc - c^2 + nc^2 + d^2 - x^2}$; or because $d^2 = b^2 - 2bc + c^2$; $u = \frac{f}{2a} \times \overline{b^2 + nc^2}$

* When the circumference has described the arc B H $= x$, it will have acquired a velocity equal to that of a body which has fallen freely from rest by the acceleration of gravity through a space $= \frac{f}{2a} \times \overline{2bx - x^2 + 2cnx - nx^2}$, as appears from the investigation in page 138; and when $x = c$, this space becomes $= \frac{f}{2a} \times \overline{2bc - c^2 + 2c^2n - nc^2} = \frac{f}{2a} \times \overline{2bc - c^2 + nc^2}$.

$\overline{-x^2}$; and the velocity of the circumference at R (fig. 4.) $=$ $\sqrt{\frac{2lf}{a}} \times \sqrt{b^2 + nc^2 - x^2}$: let t be the time of describing QR; then $\dot{t} = \sqrt{\frac{a}{2lf}} \times \frac{-\dot{x}}{\sqrt{b^2 + nc^2 - x^2}}$, and $t = \sqrt{\frac{a}{2lf}} \times$ into a circular arc of which the cosine is $\frac{x}{\sqrt{b^2 + nc^2}}$ to radius $= 1$, which should $= 0$ when $x = d$; wherefore t or the time of describing QR $= \sqrt{\frac{a}{2lf}} \times$ into a circular arc of which the cosine is $\frac{x}{\sqrt{b^2 + nc^2}} - \sqrt{\frac{a}{2lf}} \times$ into a circular arc of which the cosine is $\frac{d}{\sqrt{b^2 + nc^2}}$; and when $x = o$, that is when the entire arc QO has been described, the time $t = \sqrt{\frac{a}{2lf}} \times$ into a circular arc of which the sine is $\frac{d}{\sqrt{b^2 + nc^2}}$.

The result of this investigation is, that the time in which the balance describes the semiarc BO, by the joint action of both springs through the arc BQ, and by the action of the balance spring only through the arc QO,

is $= \sqrt{\frac{2a}{lf \times \overline{n+1}}} \times$ an arc, of which the sine is $\sqrt{\frac{c \times \overline{n+1}}{2b + 2nc}}$

$+ \sqrt{\frac{a}{2lf}} \times$ an arc, of which the sine is $\frac{d}{\sqrt{b^2 + nc^2}}$

(to radius $= 1$) expressed in parts of a second.*

* When $n = 0$, this expression becomes $\sqrt{\frac{a}{2lf}} \times$ twice an arc, of which the sine is $\sqrt{\frac{c}{2b}}$ + an arc, of which the sine is $\frac{d}{b}$: but since $c = b - d$, the two arcs here mentioned will be exactly $= 90° = \frac{p}{2}$; and the time of a semivibration $= \sqrt{\frac{a}{2lf}} \times \frac{p}{2}$ $= \sqrt{\frac{ap^2}{8lf}}$, agreeing with the solution in page 126. Suppose $d = 0$; since in this case $b = c$, the time of a semivibration becomes $\sqrt{\frac{2a}{lf \times \overline{n+1}}} \times$ an arc, of which the sine is $\sqrt{\frac{b}{2b}}$ which arc is $= 45°$, or $\frac{p}{4}$; wherefore the time of a semivibration in this case $= \sqrt{\frac{ap^2}{8lf \times \overline{n+1}}}$, which is the true value, according to the solution in page 126. See also page 146.

This solution is confined to that case in which the point of quiescence Q (fig. 4.) of the auxiliary spring is situated in the first semiarc of the vibration, that is, between B and O. Another case still remains to be considered, which is, when the point of quiescence Q of the auxiliary spring deviates from O by the given angular distance O Q, but is situated in the latter semiarc of the vibration (fig. 5.), between O and E, instead of between O and B, as in the former solution. According to this condition, making $BQ = c$, $BO = b$, and the other notation remaining as before, it appears from an investigation no ways differing from the preceding, that the time in which the balance describes the semiarc B O will be

$t = \sqrt{\frac{2a}{lf \times \overline{n+1}}} \times$ into a circular arc, of which the sine is $\sqrt{\frac{b \times \overline{n+1}}{2b + 2nc}}$ expressed in parts of a second.

This result expresses the time in which the balance describes the semiarc B O, (fig. 5.) by the accelerative force of two springs, namely, the balance spring, of which the point of quiescence is O, and an auxiliary spring, of which the point of quiescence is Q. On considering this case more fully, when applied to the actual vibrations of a balance, it will appear evident, that the action of a third spring on the balance while it is describing the semiarc BO, must be taken into the calculation, in addition to the two springs already mentioned, in order to obtain a solution entirely correspondent with the circumstances of the case, when the points of quiescence of the auxiliary springs are situated in the latter semiarcs of the vibrations.

To state this more clearly, it is to be observed, that when the points of quiescence of the balance and auxiliary springs

are coincident; the auxiliary spring commencing its action from the extremity B of the arc BO, (fig. 6.) continues to accelerate the balance till it arrives at the quiescent position O, at which point the action of the auxiliary spring entirely ceases; on this account it is plain that another auxiliary spring, equal and similar to the former, having also the point of quiescence coincident with O, must be applied to act by retardation on the balance while it describes the arc OE, in order that the times of describing the arcs BO and OE, as well as these arcs themselves, may be equal. According, therefore, to this disposition of the auxiliary springs, the balance will describe each semiarc of its vibrations precisely in the same manner as if it was impelled by one spiral spring only, the strength of which is equal to that of the balance and either auxiliary spring, when wound to the same tension. Suppose the balance to vibrate from B to E, and, for the sake of distinction, let the auxiliary spring which accelerates the balance from the extremity of the arc B, in the direction BO, be called u; and let the other auxiliary* spring which retards the balance in the semiarc OE be denoted by v; consequently, when the balance vibrates in the contrary direction from E to B, the auxiliary spring v will accelerate the balance from E to O, and the other auxiliary spring u will retard it from O to B. In respect, therefore, to the spring u, BO is the first semiarc, and OE is the latter semiarc of vibration; and on a similar principle in respect to the spring v, EO is the first semiarc, and OB is the latter semiarc of vibration.

* The circular arcs which are drawn interior to the circumference of the balance in the figures 4, 5, and 6, are intended to represent those portions of the balance's vibration in which the auxiliary springs respectively act.

In the preceding pages, the time has been investigated in which a balance vibrates in the semiarc B O, (fig. 4.) when the point of quiescence Q, of the auxiliary spring *u*, is situated between B and O, that is in the first semiarc of vibration. In this case, in order that the time of describing the latter semiarc O E, by retardation, may be equal to that of describing the first semiarc B O by accelerated motion, the quiescent point of the auxiliary spring *v* must be placed at N, between O and E, so that E N may be equal to B Q; according to this disposition of the auxiliary springs, the point of quiescence Q of the auxiliary spring *u* will be situated in the first semiarc of vibration, considering the balance as vibrating from B to E; and the point of the quiescence N of the auxiliary spring *v*, will be in the first semiarc of vibration, considering the balance as vibrating from E to B.

The third case which remains to be more fully investigated is, when the points of quiescence of the auxiliary springs are situated in the latter semiarcs of the respective vibrations; that is, when the quiescent point Q of the auxiliary spring *u* is situated in the latter arc O E, (fig. 5.) while the balance vibrates from B to E, and the point of quiescence N of the auxiliary spring *v* is situated in the latter arc O B, while the balance vibrates from E to B. In this case it is evident, that while the balance vibrates from B to O, it describes the arc B N by the acceleration of two springs; namely, the balance spring, of which the point of quiescence is O, and the auxiliary spring *u*, of which the point of quiescence is Q. And the balance describes the arc N O by the combined action of three springs; namely, by that of the balance spring, of which the point of quiescence is O, that of the auxiliary spring *u*, of

which the point of quiescence is Q, both of which accelerate the balance, and that of the auxiliary spring v, of which the point of quiescence is N, acting by retarding the balance.

The time in which the balance describes the semiarc B O, by the combined action of these three springs, will be obtained by the following investigation.

Resuming the former notation, let DO (fig. 5.) $= a = Qd = Ne$, $BO = b$, $BQ = c$, $OQ = ON = d = c - b$. The accelerative force of the balance spring on the circumference of the balance at the tension $OD = f$; the accelerative force of the auxiliary springs at the same tension Qd or $Ne = nf$. From page 138 and 139, it appears that the time in which the balance describes the arc BN by the joint action of the balance spring and auxiliary spring $u = \sqrt{\frac{2a}{lf \times \overline{n+1}}} \times$ a circular arc of which the sine $= \sqrt{\frac{\overline{b-d} \times \overline{n+1}}{2b + 2nc}}$; and the velocity acquired by the circumference at N is equal to that of a body which has fallen freely from rest by the acceleration of gravity through a space* $= \frac{f}{2a} \times \overline{b^2 - d^2 + nb^2 - 3nd^2 + 2nbd}$. To find the time of describing the arc NO, suppose the balance to have proceeded in its vibration from N to R, (fig. 5.) and make $NR = x$.

* In the investigation, page 138, it is shewn, that when the balance has described the arc $BH = x$, the space through which a body must fall freely from rest to acquire the velocity of the circumference at the point H, or $u = \frac{f}{2a} \times \overline{2bx - x^2 + 2cnx - nx^2}$, the expression being the same, whether the point Q is on one side of O or on the other, provided $BQ = c$. In the present case, $c = b + d$, and when the circumference has described the arc B N, $x = b - d$; wherefore if $b + d$ is substituted for c, and $b - d$ for x in the equation, $u = \frac{f}{2a} \times \overline{2bx - x^2 + 2cnx - nx^2}$, the result will be $u = \frac{f}{2a} \times \overline{b^2 - d^2 + nb^2 - 3nd^2 + 2nbd}$.

The accelerative force of the balance spring on the circumference of the balance at the point R is $+\frac{f}{a}\times\overline{d-x}$

The accelerative force of the auxiliary spring u, at the tension or distance from quiescence $Qd=a$ being nf, the force of the same spring at the tension or distance $QR=2d-x$, is - - - - - $+\frac{nf}{a}\times\overline{2d-x}$

The force of the auxiliary spring v, at the tension $Ne=a$ being nf, at the tension NR, the force of this spring acting by retardation will be - - - - $-\frac{nf}{a}\times x$.

Sum of the forces acting on the circumference of the balance, when it has described the arc $NR=$ - - $\frac{f}{a}\times\overline{d-x+2dn-2nx}$.

Let u be the space through which a body falls freely from rest by the acceleration of gravity, to acquire the velocity of the circumference at R; the principles of acceleration give this equation; $\dot{u}=\frac{f}{a}\times\overline{d\dot{x}-x\dot{x}+2dn\dot{x}-2nx\dot{x}}$; and taking the fluents so that when $x=o$, u may be $=\frac{f}{2a}\times\overline{b^2-d^2+nb^2-3nd^2+2nbd}$; $u=\frac{f}{2a}\times\overline{b^2-d^2+nb^2-3nd^2+2nbd+2\overline{d+4nd}\times x-\overline{1+2n}\times x^2}$; if t is put to represent the time of describing the arc NR, $\dot{t}=\sqrt{\frac{a}{2lf}}\times\frac{\dot{x}}{\sqrt{b^2-d^2+nb^2-3nd^2+2nbd+2\overline{d+4nd}\times x-\overline{1+2n}\times x^2}}$, and taking the fluents so that when $x=o$, $t=o$, and making $x=d$, the time of describing the arc $NO=\sqrt{\frac{a}{2lf\times\overline{1+2n}}}\times$ a circular arc of which the sine $=\sqrt{\frac{d^2\,\overline{1+2n}}{b^2+n\times\overline{b+d}^2-2nd^2}}$, or because

$b + d = c$, $t = \sqrt{\frac{a}{2lf \times \overline{1 + 2n}}}$ × a circular arc of which the sine $= \sqrt{\frac{d^2 \times \overline{1 + 2n}}{b^2 + nc^2 - 2nd^2}}$.

The result is, that when the points of quiescence of the auxiliary springs are situated in the latter semiarcs of the respective vibrations, the time in which the balance describes the semiarc B O will be $= \sqrt{\frac{2a}{lf \times \overline{n + 1}}}$ × a circular arc of which the sine is $\sqrt{\frac{\overline{b - d} \times \overline{n + 1}}{2b + 2nc}} + \sqrt{\frac{a}{2lf \times \overline{1 + 2n}}}$ × a circular arc of which the sine is $\sqrt{\frac{d^2 \times \overline{1 + 2n}}{b^2 + nc^2 - 2nd^2}}$.

The balance of Mr. MUDGE's time-keeper describes the semiarc B O, by the joint action of two springs, *i. e.* the balance* spring, and an auxiliary spring; each spring is wound through the same arc B O, and comes to the same point of quiescence O (fig. 6.); consequently the action of the two springs is the same with that of a single spring of equal strength with both. In this case, the time of a semivibration through the arc B O, will be obtained from a former solution; for referring to page 126, and making O D $= a$, and the force of the balance spring and auxiliary spring at the distance from quiescence O D, $= f + nf = \mathrm{F}$; the time of a semivibration is

$$= \sqrt{\frac{ap^2}{8lf \times \overline{n + 1}}} = \sqrt{\frac{ap^2}{8l\mathrm{F}}}.$$

But if the points of quiescence of the balance and auxiliary springs instead of coinciding, according to the principle of Mr. MUDGE's construction, should deviate from this adjustment by a small arc O Q = O N (fig. 4 and 5.); to what extent the

* A double spiral spring is applied in the balance of Mr. MUDGE's time-keeper, but as these two springs act as one spring, they are here considered as such.

daily rate of the time-keeper may be affected by this alteration in the position of the quiescent points, remains unknown, unless it be investigated by assuming the deviation of the points of quiescence, as one of the conditions on which the time of vibration depends. This condition is included in the preceding investigations, which may be now applied to the solution of some cases, which are suggested by considering the construction of Mr. MUDGE's time-keeper.

Any minute or particular description of this ingenious invention would be foreign to the subject of theoretic investigation; an outline only of the construction on which the action of the several springs depends will be necessary, to render the application of the preceding theorems sufficiently intelligible.

ONEBQ (Tab. XV. fig. 8.) is the circumference of the balance, vibrating by the action of a spiral spring on an axis C A D H, passing through the centre C; the axis is discontinued from A to D to make room for the other parts of the work. C A and D H are connected by means of a branch or crank A X Y D, which is fixed to the axis C A D H, and always vibrates with the balance on the said axis.*

L M, Z W, are two rods affixed to the crank at the points L and Z parallel to X Y; which rods also vibrate with the balance. *c, d, e, f, g, r, s,* are fixed parts of the machine. T R is an axis in the same right line with C A D H carrying an arm G O at right angles to it † (or nearly so), and a small auxiliary spring *u,* which is wound up whenever the arm G O is turned round

* The additional weight affixed to the balance at *t* counterpoises the weight of the crank or branch A X Y D, so as to bring the centre of gravity of the whole into the axis of motion.

† In the machine G O and I O are not exactly at right angles to T R and F S; but are so represented in the figure, in order to make the different positions of the arms G O and I O the more distinct.

the axis T R in the direction of the arc O *b*; *p* is a curved pallet fixed to the axis T R, which receives the tooth of the balance wheel near the axis; the tooth proceeding along the curved surface by the force of the main spring, turns the axis and the annexed arm G O in the direction of the arc O *b*, and at the same time winds up the auxiliary spring *u*. A small projection at the extremity of the curved surface of the pallet *p* prevents further progress of the tooth, when the arm O G has been turned through an arc O *b** of about 27°; consequently the spring *u* has then been wound through the same arc or angle O G *b* = 27°.

F S is another axis in the same right line with C A D H, exactly similar to that which has been described. This axis F S carries with it the arm I O; the auxiliary spring *v* is annexed to this axis, and is wound up when the axis is turned in the direction of the arc O *k*. *q* is a pallet similar to the former pallet *p*, and is placed so as to receive the tooth of the balance wheel, which by its action on the pallet winds up the spring *v*, and carries the arm O I through an angle O I *k*, = 27°, further motion being prevented by a small projection at the extremity of the pallet *q*, similar to that which has been already mentioned. *l m* represents the balance wheel, the upper tooth of which acts on the pallet *p*, and the lower tooth on the pallet *q*, alternately winding up the auxiliary springs *u* and *v*, in the manner described in the subsequent page: the axis of the balance wheel *n o* is parallel to the line C O or G O.

The several arcs expressed on the circumference of the balance, *i. e.* O Q, O *b*, O N, O *k*, are equal to the respective arcs denoted by the same letters in the circumference of the circles described by the extremities of the arms G O and I O.

* *Vide infra* the note in page 150.

Suppose that when the balance is quiescent, the main spring being unwound, the branch or crank A X Y D is in the position represented in fig. 8, A X being parallel to C O; if the quiescent points of the auxiliary springs coincide with that of the balance spring, the arm G O will just touch the rod L M; and in like manner the arm I O will just touch the rod W Z; the two arms G O and I O, in this position, are parallel to the line C O. This position of the balance and auxiliary springs remains as long as the main spring of the machine continues unwound; but whenever the action of the main spring sets the balance wheel in motion, a tooth thereof meeting with one or other of the pallets p or q, will wind up one of the auxiliary springs; suppose it should be the auxiliary spring u: the arm G O being carried into the position G b, by the force of the balance wheel acting on the pallet p, remains in that position as long as the tooth of the balance wheel continues locked by the projection at the extremity of the pallet p: and the balance itself not being at all affected by the motion of the arm G O, nor by the winding up of the spring u, remains in its quiescent position; consequently no vibration can take place except by the assistance of some external force to set the machine in motion. Suppose an impulse to be given to the balance sufficient to carry it through the semiarc O B, which is about* 135°, according to Mr. Mudge's construction.

The balance during this motion carries with it the crank A X Y D, and the affixed rods L M, Z W. When the balance has described an angle of about* 27° = to the angle O C b or O G b, the rod L M meets with the arm G b, and by turning the axis T R, and the pallet p, in the direction of the arc O b, releases the tooth of the balance wheel from the pro-

* *Vide infra* the note in page 150.

jection at the extremity of the pallet *p*; the balance wheel immediately revolves, and the lower tooth meeting with the pallet *q* winds up the auxiliary spring *v*, and carries the arm I O with a circular motion through the angle O I *k* about 27°; in which position the arm I O remains as long as the tooth of the balance wheel is locked by the pallet *q*. While the spring *v* is winding up through the arc O *k*, the balance describes the remaining part of the semiarc *b* B, and during this motion the rod L M carries round the arm G *b*, causing it to describe an angle *b* C B, or *b* G B*, which is measured by the arc *b* B = †108°: when the balance has arrived at the extremity of the semiarc O B = 135°, the auxiliary spring *u* will have been wound up through the same angle = 135°, *i. e.* 27°, by the force of the main spring acting on the pallet *p*, and 108° by the balance itself carrying along with it the arm G O or G *b*, while it describes the arc *b* B. The balance therefore returns through the arc B O by the joint action of the balance spring, and the auxiliary spring *u*; the acceleration of both springs ceasing the instant the balance arrives at the quiescent point O; when the balance has proceeded in its vibration about 27° beyond the point O to the position C *k*, the rod Z W meets with the arm I *k*, and by carrying it forward releases the tooth

* The angle *b* G B is not expressed in the figure for want of room, but is easily imagined, being precisely equal to the angle *b* C B represented on the balance.

† This magnitude of the arc *b* B has been inferred from the following circumstances communicated by Mr. MUDGE. 1st. " The whole arc of vibration is about $\frac{3}{4}$ of " a circle, or 270°, consequently the semiarc of vibration is 135°. 2dly. The ac- " tion of the balance by carrying round the arm O G from *b* to B winds up the auxi- " liary spring through an angle about four times greater than the angle O C *b* through " which it is wound up by the action of the balance wheel on the pallet." Wherefore if O B, or 135°, is divided into five parts, one of them will be O *b* = 27°, and the other four parts, or *b* B = 108°.

of the balance wheel from the pallet q; the balance wheel accordingly revolves, and the upper tooth meeting with the pallet p, winds up the auxiliary spring u as before. The balance with the crank proceeding to describe the remaining semiarc k E, winds up the spring v through the further angle k C E $= 108°$; the balance returns through the semiarc E O, by the joint action of the balance spring and the auxiliary spring v, both of which cease to accelerate the balance the instant it has arrived at O.

It is remarkable, according to this construction, that no force or impulse whatever is communicated to the balance from the main spring, and yet the vibrations are continued of their due length: on further consideration it appears that the maintaining power of the machine, instead of communicating any force or impulse, acts by removing a part of the force which retards the balance while it is describing the latter semiarc of each vibration.

In the preceding account it has been shewn, that the balance describes the semiarc from B to O by the joint action of the two springs; now for a moment let it be supposed that the balance vibrated through the entire arc B O E (fig. 7 and 8.) by the joint action of the two springs, in the same manner as if one balance spring only was applied of the same strength with both; in this case, the balance commencing its vibration at the extremity of the arc B, after having passed the semiarc B O with an accelerated motion, would describe an equal arc O E on the other side of O, by retarded motion, provided it was not obstructed by friction or other irregular resistances; but such resistances taking place will cause the latter semiarc, which is described by retarded motion, to fall short of the arc O E by some small difference E S. There are two modes by which this latter semiarc may be restored to its due length O E; either by communicating an impulse to the balance from the

main spring, or by removing a part of the force which retards the ascent of the balance while it describes the latter semiarc O E. Mr. MUDGE has discovered and applied to his time-keeper the latter mode of supplying the power which is required to continue the balance in motion. During the progress of the balance through the semiarc B O, it is accelerated by the joint action of the balance spring and auxiliary spring *u*; but while it describes the latter semiarc O E (fig. 7 and 8.), it is retarded by the joint actions of the balance spring and auxiliary spring, only while it describes a part of the semiarc from *k* to E, the retardation of the auxiliary spring being removed while it describes the first 27° of this semiarc from O to *k*.

It is evident, according to Mr. MUDGE's construction, that the diminution of retardation, which is equivalent to the supply of power to the balance, must always be of the same magnitude, so far as regards the influence of the main spring, provided it has sufficient force to wind up the auxiliary springs through the constant angle O G *b* = O C *k* (fig. 8.); and the effects of heat and cold on the auxiliary springs are necessarily included in the compensation which is applied for heat and cold to the balance spring. The maintaining power therefore, by which the motion of the balance is continued, must be always uniformly the same; this is an object usually held to be of material consequence in the construction of watches, and though often attempted by ingenious persons, has probably been accomplished in its full extent for the first time by Mr. MUDGE.

This construction possesses the further advantage of having the balance perfectly detached from the wheel work of the machine; the only communication* between the balance and the

* The exact times in which the balance describes any portions of the arc of vibration may be readily obtained by having recourse to the theorem investigated in page

balance wheel is that which subsists while the pallet is disengaged from the tooth; an instant of time in a practical sense almost evanescent: it should also be remarked, that the pressure of the tooth against the projection of the pallet is diminished before the tooth is released, by the counteraction of the other pallet on the opposite tooth of the balance wheel; which action takes place just before the tooth of the balance wheel first mentioned is released from the pallet: by this diminution of pressure, the tooth is the more easily detached from the projection of the pallet. These circumstances concur in giving to the machine facility and lightness of motion.

The preceding investigations are in the next place to be ap-

126. If we can ascertain the arc described by the balance while the tooth of the balance wheel is released from the projection of the pallet, the portion of time in which the balance is connected with the wheel-work of the machine will be known. By an experiment made on a very exact model of Mr. MUDGE's construction, it appeared that the balance described an arc of about 8°, while the tooth was released from the pallet. And since the balance describes 108° of its semivibration before the pallet begins to be moved, it will have described 116° before the tooth is released; we are therefore to find the times in which the balance describes severally the two arcs 108° and 116°: the difference of these times will be the time in which the balance wheel is released from the pallet. By the theorem referred to

	Parts of a second.
The time in which the balance describes an arc of 116° is -	0.09101
Time of describing an arc of 108° is - - -	0.08718
The difference, or time during which the balance is connected with the other parts of the machine, - - - -	0.00383

which is about $\frac{1}{260}$ part of a second of mean time, the effects of friction not being here considered.

If therefore the time of a semivibration is supposed to be divided into 100 equal parts, or physical instants, during the first 91 of them the balance will describe 108° from the extremity of the arc of vibration, to the contact of the arm connected with the pallet. During the 4 succeeding instants the pallet will be released, and during the 5 instants following, the balance will describe the remaining arc of the semivibration. Thus it appears, that the balance moves perfectly free and unconnected with the wheel-work of the machine during 96 parts in 100 of the time of each vibration.

plied to ascertain what alteration of the daily rate is occasioned in consequence of any deviation from coincidence of the points of quiescence of the balance spring and auxiliary springs, (fig. 8, and fig. 4 and 5.) that may have taken place either through a casual derangement of their position, or from a purposed adjustment. This inquiry offers three cases, which are to be separately considered; the first is, when the points of quiescence of the auxiliary springs coincide with the point of quiescence of the balance spring. 2dly. When the point of quiescence Q of the auxiliary spring *u* is situated in the first semiarc of the vibration, or between B and O (fig. 4 and 8.). 3dly. When Q is situated in the latter semiarc of vibration between E and O (fig. 5.); the like cotemporary position is to be understood in respect of the other auxiliary spring.

The consideration of these cases requires that the accelerative* forces of the balance spring and auxiliary springs, when at a given tension, should be known. This point will not be difficult to ascertain, if the proportion of the spring's forces at the same tension is given. By information received from Mr. MUDGE, it appears that the force of each auxiliary spring is about $\frac{1}{20}$ part of the force of the balance spring. Assuming this proportion, let the points of quiescence of the auxiliary springs be supposed to coincide with the point of quiescence of the balance spring, and let the radius of the balance be taken at one inch; the watch being adjusted to mean time, when the balance makes five vibrations in a second, if the force of gravity is $= 1$, the force which accelerates the circumference at an

* The accelerative force being always proportional to the absolute forces of impulse (as measured or counterpoised by a weight) divided by the mass moved equivalent to the resistance of inertia, the latter quantity remaining the same in the case of a balance's vibration, the accelerative forces will be in the same ratio with the forces of impulse.

angular distance of 90° from the quiescent point will be = 1.00408926 (p. 129): this force, in the present instance, is compounded of the force of the balance spring, and that of the auxiliary spring, which are assumed to be in the proportion of 20 to 1; consequently the force of the balance spring to accelerate the circumference at the tension 90°, or $f =$ 0.9562754
and the force of the auxiliary spring at the same tension, or $\frac{1}{20}$ of $f =$ - - - .0478138

Joint force of both springs to accelerate the circumference at the tension, 90° or $f + \frac{1}{20}f =$ 1.0040892

With* this force the balance vibrates five times in one se-

* From the investigation in page 128 it appears, that if F is the accelerative force on the circumference of the balance at the angular distance from quiescence $c°$, $p =$ 3.14159, &c. $r =$ the radius of the balance expressed in inches, $l = 193$ inches; the time of a semivibration will be $= \sqrt{\frac{p^3\, r\, c^\circ}{8\, l\, F \times 180^\circ}}$.

In the present case $c° = 90°$, $r = 1$ inch $F = f + \frac{1}{20}f = 1.0040892$; wherefore the time of a semivibration $= \sqrt{\frac{p^3\, r\, c^\circ}{8\, l\, F \times 180^\circ}} = \frac{1}{10}$ part of a second precisely.

If the balance should vibrate by the force of the auxiliary springs only, the force of acceleration on the circumference at the distance 90° from quiescence is $\frac{1}{20} f =$ 0.0478138; wherefore $\frac{1}{20}f$ being substituted for F in the quantity $\sqrt{\frac{p^3\, r\, c^\circ}{8\, l\, F \times 180^\circ}}$ the other values remaining the same as in the former case, the time of a semivibration will now be $\sqrt{\frac{p^3\, r\, c^\circ}{8\, l \times \frac{1}{20} f \times 180^\circ}} =$ pts. of a sec. .45825; and the time shewn by the watch in any portion of mean time t will $= \frac{t}{4.5825} = t \times .21822$. Thus if t should be taken = to one minute, or 60 seconds of time, the time shewn by the watch in 60 seconds will be $= 60 \times .21822 = 13$ seconds nearly.

This would be the result if the forces of the balance and auxiliary springs were precisely in the proportion of 20 to 1. Mr. MUDGE mentioned this proportion by esti-

cond when adjusted to mean time; the daily rate will therefore in this case be = 0.

Now let Q (fig. 4 and 8.), or the point of quiescence of the auxiliary spring deviate from O, the point of quiescence of the balance spring, by an arc O Q; suppose this arc O Q to be = 1°; and let the point Q be situated in the first semiarc of vibration between O and B. The time of describing the semiarc B O will be ascertained on these conditions, by referring to the investigation (page 140), and making the following substitutions:

$$a = 1.5707963 = \text{an arc of } 90^\circ \text{ to radius} = 1\ *$$
$$b = 2.3561945 = \text{an arc of } 135^\circ$$
$$c = 2.3387412 = \text{an arc of } 134^\circ$$
$$b - c = d = 0.0174533 = \text{an arc of } 1^\circ$$
$$l = 193$$
$$f = 0.9562754$$
$$n = \tfrac{1}{20}.$$

mation only, not having any memorandum of experiments made to ascertain the exact proportion.

If therefore the actual rate of the watch should be observed when the balance vibrates by the action of the auxiliary springs only, it is probable that the time shewn by the watch in 60 seconds of mean time might differ somewhat from that which has been here stated.

Since these notes were written, I have been favoured by his Excellency Count BRUHL with an account of an observation made on the rate of Mr. MUDGE's first time-keeper, when the balance vibrated by the action of the auxiliary springs only, the balance spring being removed. According to this observation, the watch shewed twelve minutes by the motion of the hands in one hour of mean time, which corresponds to an interval of 12 seconds of time, shewn by the watch in 60 seconds, or one minute of mean time. According to the calculation, 13 seconds of time are shewn by the watch in one minute. A nearer agreement between the theory and matter of fact could scarcely be expected in the circumstances of the experiment.

* In all the following calculations the radius is also = 1.

Then, $\sqrt{\dfrac{2a}{lf \times \overline{n+1}}} \times$ into a circular arc, of which the sine is

$\sqrt{\dfrac{c \times \overline{n+1}}{2b+2nc}} =$ - - - parts of a second, 0.09955070

$\sqrt{\dfrac{a}{2lf}} \times$ into a circular arc, of which the sine

is $\dfrac{d}{\sqrt{b^2+nc^2}} =$ - - - 0.00047174

Time of a semivibration in the arc BO = - .10002244

The time shewn by the watch in $24^h = \dfrac{24^\circ}{1.0002244} = 23^h\ 59'$ $40''.60$, giving a daily rate of $19''.40$ slow.

This variation of the daily rate is not to be considered as affecting the regularity of the watch, as it is either compensated by adjustments when the watch is regulated to mean time, or taken as the established rate. A more material point is next to be determined; admitting the deviation of O and Q (fig. 4. and 8.) to be OQ = 1°, the same as in the former case; suppose the semiarc of vibration to be diminished from 135° to 125°. If the points of quiescence O and Q were coincident, this diminution of the arc of vibration would cause no alteration (page 128) in the time of a semivibration, because the forces of acceleration of both springs would be as the angular distances from the same quiescent point O; but since these points are separated by an arc of 1°, a diminution of 10° in the semiarc of vibration, will cause a change in the daily rate, which will be obtained from the general theorem (page 140) by making the following substitutions:

$$a = 1.5707963 = \text{an arc of } 90^\circ$$
$$b = 2.1816616 = \text{an arc of } 125^\circ$$
$$c = 2.1642083 = \text{an arc of } 124^\circ$$
$$b - c = d = 0.0174533 = \text{an arc of } 1^\circ.$$

$$l = 193$$
$$f = 0.9562754$$
$$n = \frac{1}{20}.$$

$\sqrt{\frac{2a}{lf \times \overline{n+1}}} \times$ into a circular arc, of which the sine is

parts of a second.

$\sqrt{\frac{c \times \overline{n+1}}{2b + 2nc}} =$ - - - 0.09951472

$\sqrt{\frac{a}{2lf}} \times$ a circular arc, of which the sine is

$\frac{d}{\sqrt{b^2 + nc^2}} =$ - - - - 0.00050949

The time of a semivibration - .10002421

Time shewn by the watch in $24^h = \frac{24^h}{1.0002421} = 23^h\ 59'\ 39''.08$

giving a daily rate of - - - $20''.92$ slow.

Daily rate, when the semiarc of vibration was 135° $19''.40$ slow.

Retardation of the daily rate in consequence of the diminution of the semiarc of vibration from 135° to 125° - - - - $1''.52$.

In these examples, the point of quiescence of the auxiliary spring Q is situated in the first semiarc of vibration between B and O; and in consequence of this position, the daily rate of the time-keeper is retarded by a separation of the points of quiescence, while the semiarc of vibration remains the same; *i. e.* 135°; secondly, the separation of the points of quiescence remaining unaltered, a diminution of the arc of vibration causes a further retardation of the rate. Opposite effects on the daily rate take place when the point of quiescence Q is situated in the latter semiarc of vibration, (fig. 5.) between O and E. The arc OQ remaining = 1° as before, this case produces an acceleration instead of a retardation of the rate.

Suppose the semiarc of vibration B O to be 135°
Separation of the points of quiescence $=$ O Q 1°

Arc B Q in fig. 5, or B N in fig. 8. - 136°

The time of describing the semiarc B O will be determined by referring to the theorem (page 146).

making $a = 1.5707963 =$ an arc of 90°
$b = 2.3561945 =$ an arc of 135°
$c = 2.3736478 =$ an arc of 136°
$d = 0.0174533 =$ an arc of 1°
$l = 193.$
$f = 0.9562754.$
$n = \frac{1}{20}.$

The time of describing the semiarc B O $= \sqrt{\dfrac{2a}{lf \times \overline{n+1}}}$

Parts of a second.

$\times$ a circular arc of which the sine is $\sqrt{\dfrac{\overline{b-d} \times \overline{n+1}}{2b + 2nc}} = 0.09950617$

$\sqrt{\dfrac{a}{2lf \times \overline{1+2n}}} \times$ a circular arc of which the sine

is $\sqrt{\dfrac{d^2 \times \overline{1+2n}}{b^2 + nc^2 - 2nd^2}}$ - - $= 0.00047141$

Time of a semivibration in the arc B O $= 0.09997758$

Time shewn by the watch in $24^h = \dfrac{24^h}{.9997758} = 24^h\ 0'\ 19''.38$, giving a daily rate of $19''.38$ fast.

Every thing else remaining, let the semiarc of vibration be diminished from 135° to 125°, and make

$b = 2.1816616 =$ an arc of 125°
$c = 2.1991149 =$ an arc of 126°

The time of describing B O will now be $\sqrt{\dfrac{2a}{lf \times \overline{n+1}}} \times$ a

Parts of a second.

circular arc of which the sine is $\sqrt{\frac{\overline{b-d} \times \overline{n+1}}{2b+2nc}} = 0,09946662$

$\sqrt{\frac{a}{2lf \times \overline{1+2n}}}$ × a circular arc of which the sine is $\sqrt{\frac{d^2 \times \overline{1+2n}}{b^2+nc^2-2nd^2}}$ - - $= .00050912$

Time of a semivibration in the arc B O $= .09997574$

Time shewn by the watch in $24^h = \frac{24^h}{.9997574} = 24^h\ 0'\ 20''.91$, giving a daily rate of - - $20''.91$ fast.

Daily rate when the semiarc of vibration was $135°$ - - - $19''.38$ fast.

Acceleration of the rate in consequence of the diminution of the semiarc of vibration from $135°$ to $125° =$ - - - - $1''.53$

The theorem from which this latter result has been calculated is founded on a supposition, that the arc B N (fig. 5.) is described by the accelerative forces of the balance spring and the auxiliary spring *u*, and that the arc N O is described by the accelerative forces of the two springs just mentioned, combined with the retarding force of the other auxiliary spring *v*, of which the point of quiescence is situated at N (fig. 5.). It may possibly be thought, that this supposed action of the auxiliary spring which retards the balance while it describes the arc from N to O, cannot take place, because by the peculiar construction of Mr. MUDGE's invention, when the balance vibrates from B towards E, the retarding force of the auxiliary spring is removed from acting on the balance while it describes the arc N $k = 27°$ (fig. 5.), including the said arc NO; but it is to be considered, that this removal of the retarding force from

acting on the balance while it describes the arc from N to k, is merely a mechanical expedient for supplying the power which is lost by friction, and as it is thus wholly employed and counteracted, it appears requisite, in making calculations of the times in which the balance vibrates, that the entire forces of the springs are to be taken into the calculation in the same manner as if friction did not exist, and no mechanical contrivance was necessary for compensating the loss of motion in consequence of it. But since, in point of fact, no retarding force acts on the balance while it describes the arc N O, according to the conditions under consideration, it may be satisfactory to calculate the time of a semivibration in the arc B O on this ground also, that is, on a supposition that the entire semiarc B O is described by the joint accelerative forces of the balance spring, and the auxiliary spring u only; it will appear that this slight variation of the conditions does not at all affect the conclusions deduced from the preceding calculations, and very little alters the results themselves.

The time of describing the semiarc BO, on these conditions, will be determined by referring to the theorem (page 141), and making the following notations.

$a = 1.5707963 =$ an arc of 90° to radius $= 1$
$b = 2.3561945 =$ an arc of 135°
$c = 2.3736478 =$ an arc of 136°
$d = 0.0174533 =$ an arc of 1°.

$$l = 193$$
$$f = 0.9562754$$
$$n = \tfrac{1}{20}.$$

The time of a semivibration is $= \sqrt{\frac{2a}{lf \times \overline{n+1}}} \times$ into a cir-

cular arc, of which the sine $= \sqrt{\frac{b \times \overline{n+1}}{2b + 2nc}} = .09997752$ pts. of a second.

Time shewn by the watch in $24^h = \frac{24}{.9997752} = 24^h\ 0'\ 19''.42$, giving a daily rate of $19''.42$ fast.

Every thing else remaining, let the semiarc of vibration be diminished from 135° to 125°; and make

$b = 2.1816616 =$ an arc of 125° to radius 1.

$c = 2.1991149 =$ an arc of 126°.

The time of describing the semiarc B O will now be $=$

$\sqrt{\frac{2a}{lf \times \overline{n+1}}} \times$ a circular arc, of which the sine $= \sqrt{\frac{b \times \overline{n+1}}{2b + 2nc}}$

$= 0.09997570$ parts of a second.

Time shewn by the watch in $24^h = \frac{24^h}{.9997570} = 24^h\ 0'\ 20''.96$,

giving a daily rate of - - - $20''.96$ fast.

daily rate, when the semiarc of vibration was $135° = 19''.42$ fast.

Acceleration of daily rate in consequence of the diminution of the semiarc of vibration from 135° to 125° - - - - $1''.54$,

scarcely differing from the former determination in page 160.

By having recourse to the figure, we may distinctly perceive the three quiescent positions of the arms G O and I O, (fig. 8.) in respect to the rods L M, Z W, which correspond with the several conditions assumed for the calculation of the preceding examples. In the first place, the crank A X Y D, and the arms G O, I O, being in their quiescent position, it is evident from the preceding observations, that when the arms G O, I O are adjusted so as just to touch the rods L M, Z W, the points of quiescence of the auxiliary springs u and v coincide with the point of quiescence of the balance spring O, (fig. 6.) and in

this case, the balance making 5 vibrations in a second when adjusted to mean time, the daily rate is = o (p. 146).

Secondly, suppose the arms G O and I O to be so affixed to the axes T R, F S, (fig. 8 and 4.) that instead of just touching the rods L M and N W when quiescent, they are inclined* to that position at any angle O G Q = O I N: the situation of the pallets *p* and *q* not being altered: in this case the point of quiescence of the auxiliary spring *u*, will be separated from the point of quiescence of the balance spring by the angular distance OGQ = OCQ. In like manner the point of quiescence of the auxiliary spring *v* will deviate from the point of quiescence of the balance spring by the angular distance O I N = O C N. Suppose that the arc OQ = ON is = 1°: the point of quiescence Q of the auxiliary spring *u* is in the first semiarc of vibration, that is, between B and O, (fig. 4.) while the balance vibrates from B to E; and the point of quiescence of the auxiliary spring *v*, is in the first semiarc of vibration between E and O, while the balance vibrates from E to B; on these conditions, the semiarcs of vibration B O and OE being 135°, the daily rate of the time-keeper will be 19″.40 slower than mean time, (p. 157) and if the semiarcs BO, O E, should be diminished from 135° to 125°, the daily rate will be 20″.92 slow (p. 158); the retardation of the daily rate in consequence of the diminished semiarc of vibration being 1″.52.

To consider the remaining case; if the arms G O, I O, (fig. 8.) are adjusted so as to press equally by the force of the auxiliary springs against the rods L M, W Z, when quiescent, the equal and contrary pressures prevent any apparent effect

* The force of the main spring not being supposed to act on the balance wheel, or pallets *p* and *q*.

on the position of the balance or crank A X Y D; if the rod LM should be removed (by turning the balance in its plane), suppose that the arm GO rests in a position GN, at a distance beyond O, which is measured by the arc O N = 1°; in this position the point of quiescence of the auxiliary spring *u* will be situated in the latter semiarc of vibration at N, between E and O, and by a similar adjustment of the quiescent position of the arm IO, (fig. 5 and 8.) the point of quiescence of the auxiliary spring *v* will be situated at Q in the latter semiarc of the vibration, between O and B; in consequence of this position of the points of quiescence, the daily rate of the watch will be accelerated 19″.38, (p. 159) while the semiarc of vibration continues 135°, and when the semiarc of vibration is diminished to 125°, the daily rate will be further accelerated; the rate being 20″.91 fast, (page 160) or 1″.53 faster than when the semiarc of vibration was 135°.

From these several* results it may be concluded, that although the rate of going of Mr. MUDGE's time-keeper depends materially on the quiescent position of the arms G O, I O, (fig. 8.), that is, on the position of the points of quiescence of the auxiliary springs, yet while that position remains unaltered, whatever it may be, the regularity of the time-keeper will not be affected in consequence of the said position, provided the semiarc of vibration continues the same; but when this arc is

* It is scarcely necessary to observe, that although these results have been investigated from supposing the elastic forces of the spiral springs to be in the precise law of the tensions, or distances from quiescence; yet if the spring's forces should deviate somewhat from that law, the general conclusions deduced from the preceding calculations, respecting the acceleration or retardation of the rate arising from an alteration in the position of the points of quiescence, will still be true, although the degree in which these effects take place may not be exactly the same as when the spring's forces are in the precise law assumed in the investigations.

diminished, an acceleration or retardation of the daily rate will take place, according to the situation of the points of quiescence of the auxiliary springs referred to the semiarcs of vibration in which these springs respectively act. If the points of quiescence of the auxiliary springs should be situated in the first semiarcs of vibration, a diminution of these arcs will cause a retardation of the rate; but if the points of quiescence should be situated in the latter semiarcs of the respective vibrations, the daily rate will be accelerated.

By means of Mr. MUDGE's construction, we may apply the principles deduced from the preceding investigations to correct such alterations in the daily rate as may arise from a diminished or increased arc of vibration. For if it should be known, from any satisfactory mode of trial, that the properties* of the balance spring are such as cause the longer arcs of vibration to be described in less time than the shorter arcs; whenever the arc of vibration is diminished, the time-keeper will lose; and this error in the rate would be corrected by adjusting the points of quiescence of the auxiliary springs in the respective latter semiarcs of vibration at such a distance ON

* Different opinions have been entertained respecting the times in which a balance, vibrating freely by the action of a spiral spring, describes the longer and shorter arcs of vibration. Mr. HARRISON says, "large arcs are naturally performed in less time than "small ones."—Notes taken at the Discovery of Mr. HARRISON's Time-keeper, p. vii. In this opinion he is followed by M. BERTHOUD; "J'ai appris par des experiences sûres, "que les grand arcs et les petits arcs d'un balancier ne sont pas isochrones, et qu'en "general, dans un balancier libre, les grands arcs sont plus prompts que les petits." Mr. LUDLAM, in his report addressed to the commissioners of the board of longitude, and intituled, a Short View of the Improvements made or attempted in Mr. HARRISON's Watch, has the following remark: "The principle on which Mr. HARRISON forms "the alteration of the third part (before described) is, that the longer vibrations of a "balance moved by the same spring are performed in less time. This is contrary to "the received opinion among philosophers and workmen."

These contradictory opinions may possibly have arisen from experiments, in which

= O Q (fig. 8 and 5.) from the point of quiescence of the balance spring, as corresponds with the error intended to be rectified.* In like manner, if the property of the balance spring should be such that a diminution of the arc of vibration causes an acceleration of the daily rate, this error will be corrected by placing the points of quiescence of the auxiliary springs in the first semiarcs of vibration, (fig. 4 and 8.) at their proper distances from the point of quiescence of the balance spring. It is, however, to be remarked, that if the balance spring should be of the latter kind here assumed, and the points of quiescence of the auxiliary springs, either by any casual derangement in their position, or by an adjustment of them for the purpose of experimental observation, should be placed in the latter semiarcs of vibration, the effect of this position would be an acceleration of the rate, whenever the semiarc of vibration is diminished: and this effect would be produced on a double account; first, from the assumed nature of the balance spring, which disposes it to describe the smaller arcs in less time

all the circumstances capable of influencing the times of vibration in longer or shorter arcs, were either not noticed, or omitted to be properly allowed for; this will seem the more probable if it be allowed, that a balance spring may be adjusted in various ways so that either the longer or shorter arcs shall be performed in the least time; not only by altering the thickness and strength of the spring in different parts, but, if we subscribe to the opinion of M. BERTHOUD, in a spring uniform in every respect throughout, by altering the length and number of turns. He infers, that a certain length and number of turns may be given to an uniform spiral spring which will make it perfectly isochronal. This latter principle, however, does not appear to have been verified by any satisfactory experiments. According to the inferences deduced from the preceding investigations, three spiral springs, which are not isochronal when acting singly, may be so united by properly adjusting their points of quiescence, that their combined action shall cause the balance to perform its vibrations in any two arcs of unequal lengths in the same time.

* The position of the points of quiescence of the auxiliary springs is here understood to be altered, by affixing the arms G O, I K, differently on the axes T R, F S; the quiescent position of the pallets being no ways changed.

than the larger arcs of vibration: and secondly, from the position of the points of quiescence of the auxiliary springs. But it is evident from the preceding considerations, that although the balance spring should not be isochronal, yet the regularity of the time-keeper will not be at all affected, however the points of quiescence of the auxiliary springs may be situated in respect of the point of quiescence of the balance spring, as long as the semiarc of vibration continues unchanged; and if the semiarc of vibration should be liable to increase or diminution, Mr. MUDGE's construction affords an effectual remedy against this cause of variation in the rate, since the arms projecting from the auxiliary springs may be so adjusted, as to place their points of quiescence either in the first or latter semiarcs of vibration, according as the balance spring, when acting singly, causes the shorter or longer vibrations to be described in the least time.

On revising the preceding pages a few observations have occurred, which may be here inserted.

Note to page 122, *line* 13.—The elastic force of the spiral spring when at a given tension to turn the balance, is here assumed to be the same, whether the balance is at rest or in motion, being in both cases equal to the weight by which the spring's force at the given tension would be counterpoised.

Note to page 131.—It is not necessary to add constant quantities to the fluents of the fluxions $\frac{-\dot{x}}{\sqrt{b^{n+1} - x^{n+1}}}$, $\frac{-\dot{y}}{\sqrt{c^{n+1} - y^{n+1}}}$; because when the entire fluents are taken, they are precisely in the same proportion, whether the constant quantities (or corrections as they are sometimes termed) are added or omitted.

Addition to the note in page 140.—If the points of quiescence are in the first semiarc of vibration, and $c = 0$, the point B will coincide with Q (fig. 4.), from which point the vibration will commence; in this case the expression for the time of a semivibration will become $t = \sqrt{\frac{a}{2lf}} \times$ an arc of which the sine is $\frac{d}{d}$, or an arc of $90° = \frac{p}{2}$, or $t = \sqrt{\frac{ap^2}{8lf}}$; which agrees entirely with the solution in page 126; for in this case the auxiliary spring not acting on the balance while it describes the arc Q O N, the balance will vibrate by the force of the balance spring only; of which the force at the distance a or 90° is $= f$, and consequently by the theorem investigated in page 126,

the time of a semivibration is $\sqrt{\frac{a p^2}{8 l f}}$ the same as is deduced from the more general expression.

Corollary to the solution in page 146.—If the points of quiescence of the auxiliary springs are in the latter semiarcs of vibration, and the vibration commences from the point N (fig. 5.), in this case $d = b$, or $c = 2d$, and by the solution in page 146, the time of a semivibration becomes $\sqrt{\frac{a}{2 l f \times \overline{1 + 2n}}}$ $\times$ an arc of which the sine is $\sqrt{\frac{d^2 \times \overline{1 + 2n}}{d^2 \times \overline{1 + 2n}}}$, or an arc of $90° = \frac{p}{2}$: wherefore the time of a semivibration $=$ $\sqrt{\frac{a p^2}{8 l f \times \overline{1 + 2n}}}$. We observe, therefore, that whether the points of quiescence of the auxiliary springs are situated in the first or latter semiarcs of vibration, if the semiarc of vibration should be $=$ to the distance of the said points from the point of quiescence of the balance spring, the times of vibration will be the same whatever be the magnitude of that semiarc.

Note to page 164.—Supposing, as in the former examples, the points of quiescence of the auxiliary springs to be at the distance of 1° from the point of quiescence of the balance spring, the variations of rate from mean time, when the semiarcs of vibration are 135°, 125°, 60°, and 10° severally, will be as expressed underneath:

Points of quiescence in the first semiarcs of vibration.

Semiarcs of vibration.	Variation of the daily rate from mean time.
135°	— 19″.40
125°	— 20″.92
60°	— 44″.33
10°	— 4′.24″.24

Points of quiescence in the latter semiarcs of vibration.

135°	+ 19″.38
125°	+ 20″.91
60°	+ 43″.60
10°	+ 4′.23″.60

If auxiliary springs should be applied for the sole purpose of making the vibrations of the balance isochronal, it may probably be found convenient to adjust the points of quiescence of the auxiliary and balance springs at a greater distance than 1°; suppose the distance to be 10°. Let the arc of vibration be 450°, or $1\frac{1}{4}$ of a revolution, and suppose the balance to make four vibrations in a second; on these conditions two auxiliary springs, each of which is about $\frac{1}{40}$ part of the force of the balance spring, or perhaps a single auxiliary spring proportionally stronger, will, in most cases, be sufficient to compensate for the want of isochronism in the balance spring.

Fig. 1.

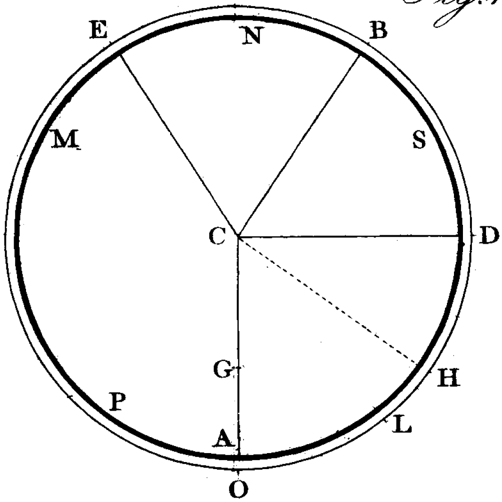

Fig. 2.

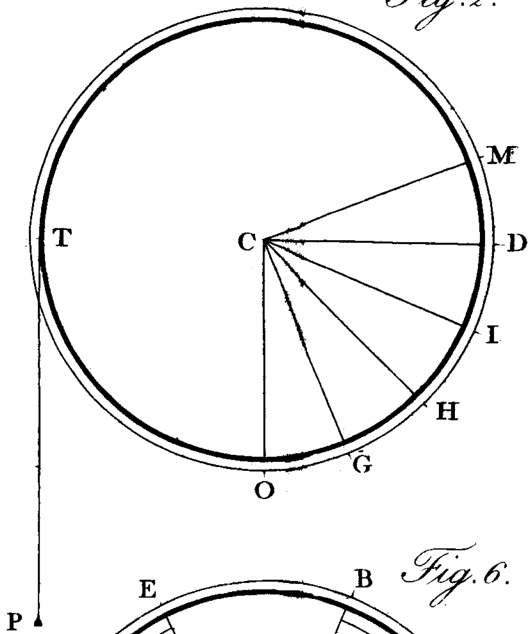

Fig. 5.

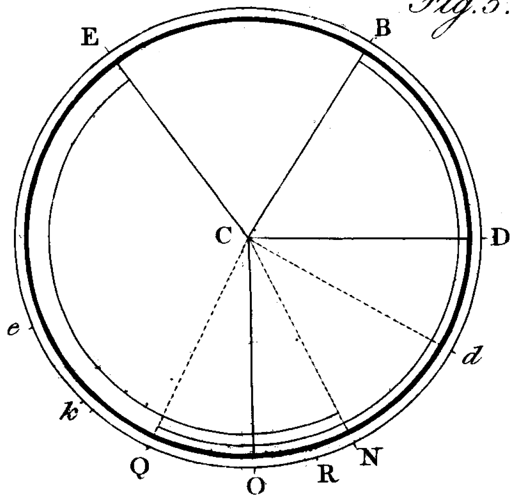

Fig. 6.

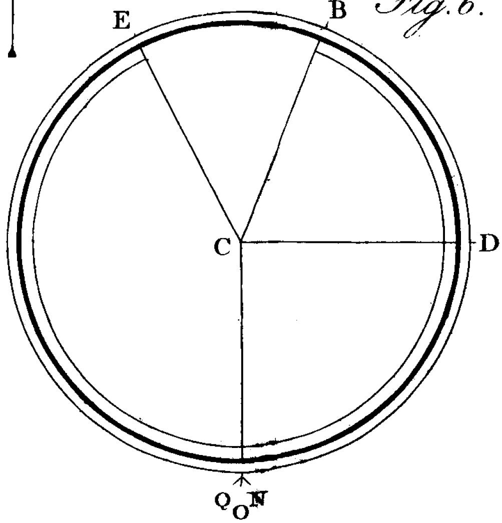

Fig. 3.

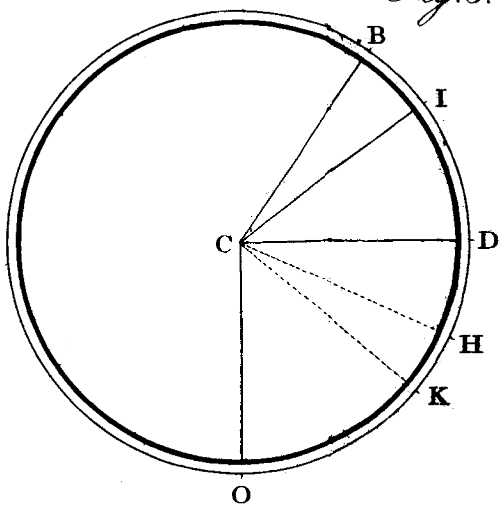

Fig. 4.

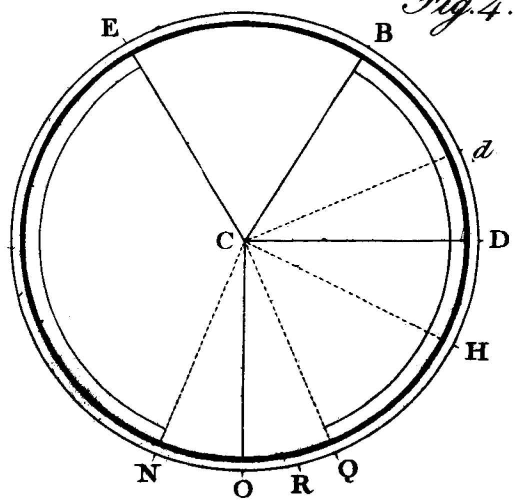

Fig. 7.

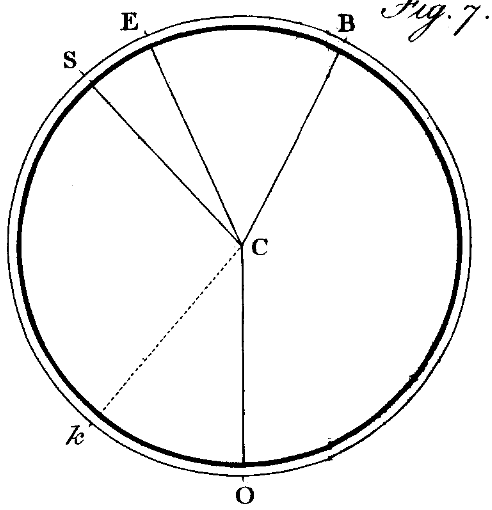

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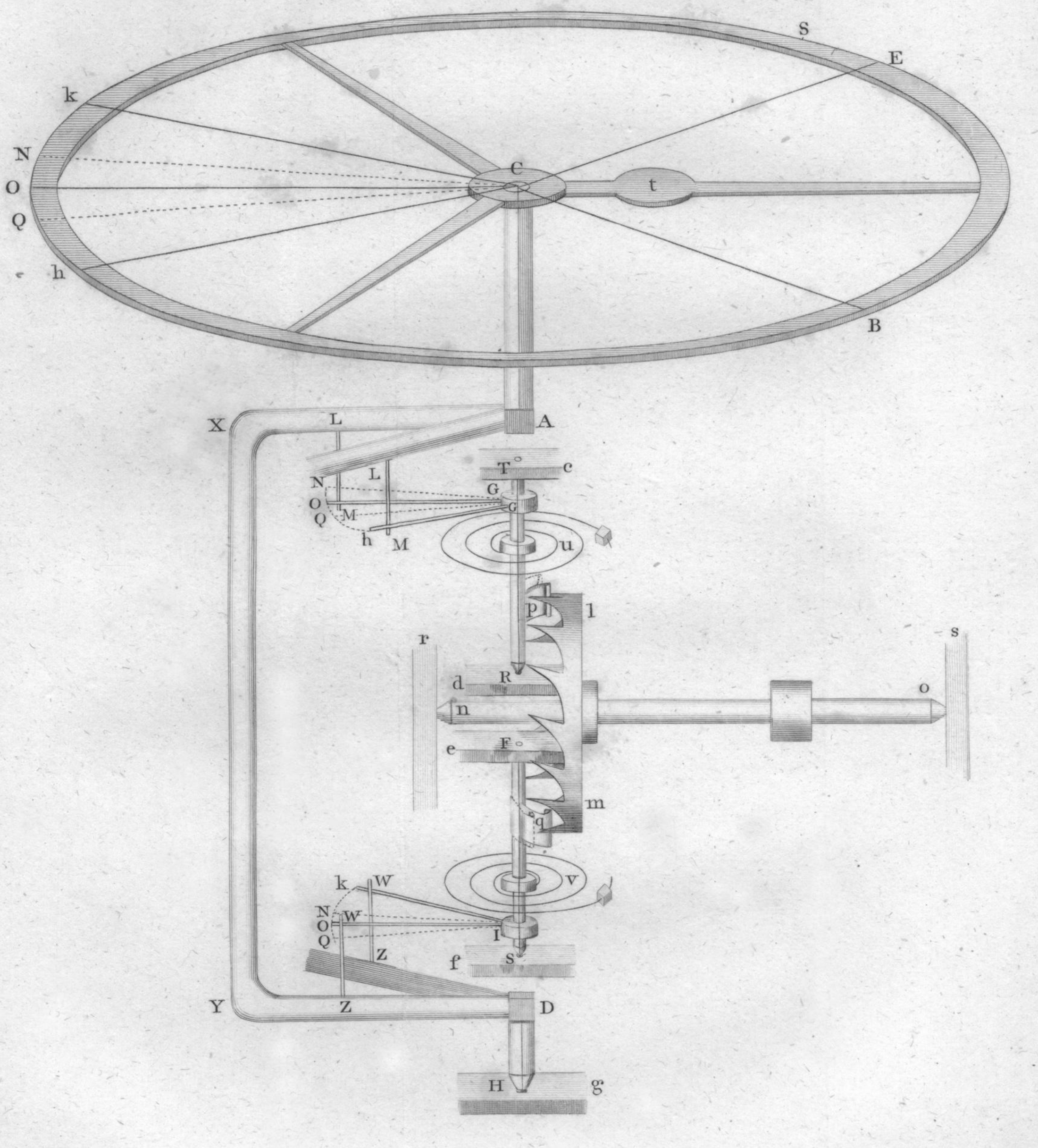
Fig. 8.



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